

ON THE SECOND HANKEL DETERMINANT OF CONCAVE FUNCTIONS

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In dedication to Professor Karl-Joachim Wirths on his 70th birthday.

ABSTRACT. In the present paper, we will discuss the Hankel determinants $H(f) = a_2a_4 - a_3^2$ of order 2 for normalized concave functions $f(z) = z + a_2z^2 + a_3z^3 + \dots$ with a pole at $p \in (0, 1)$. Here, a meromorphic function is called concave if it maps the unit disk conformally onto a domain whose complement is convex. To this end, we will characterize the coefficient body of order 2 for the class of analytic functions $\varphi(z)$ on $|z| < 1$ with $|\varphi| < 1$ and $\varphi(p) = p$. We believe that this is helpful for other extremal problems concerning a_2, a_3, a_4 for normalized concave functions with a pole at p .

1. INTRODUCTION

A meromorphic function f on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ of the complex plane \mathbb{C} is called *concave* if f is univalent and if $\mathbb{C} \setminus f(\mathbb{D})$ is convex. Such functions are intensively studied by Avkhadiiev, Bhowmik, Pommerenke, Wirths and others in recent years, see [1, 2, 3, 4, 6]. For $p \in \mathbb{D} \setminus \{0\}$, we denote by $\mathcal{C}o_p$ the set of concave functions f with a pole at p normalized by $f(0) = 0$ and $f'(0) = 1$. By a suitable rotation, we will assume without loss of generality that $0 < p < 1$ in what follows. Each function f in $\mathcal{C}o_p$ can be expanded in the form $f(z) = z + a_2z^2 + a_3z^3 + \dots$ for $|z| < p$. We sometimes write $a_n = a_n(f)$ to indicate that the coefficients belong to the function f .

By $\text{End}(\mathbb{D})$ we denote the set of analytic endomorphisms (self-maps) of the unit disk \mathbb{D} . Let \mathcal{B}_p stand for the class of $\varphi \in \text{End}(\mathbb{D})$ fixing the point p . The first author gave the following characterization of the functions in $\mathcal{C}o_p$ in [10].

Theorem A. *Let $0 < p < 1$. For $f \in \mathcal{C}o_p$, there exists a $\varphi \in \mathcal{B}_p$ such that*

$$(1.1) \quad f'(z) = \frac{p^2}{(z-p)^2(1-pz)^2} \exp \int_0^z \frac{-2\varphi(t)}{1-t\varphi(t)} dt, \quad z \in \mathbb{D}.$$

Conversely, for a given $\varphi \in \mathcal{B}_p$, there exists a function $f \in \mathcal{C}o_p$ satisfying (1.1).

We remark that the condition $\varphi(p) = p$ comes from the demand that $f'(z)$ should have no residue at $z = p$.

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For $a \in \mathbb{D}$, the Möbius transformation

$$T_a(z) = \frac{a - z}{1 - \bar{a}z} = -[z, a]$$

is an analytic involution of \mathbb{D} interchanging 0 and a . Here $[z, w] = (z - w)/(1 - \bar{w}z)$ denotes the *complex pseudo-hyperbolic distance* introduced by Beardon and Minda [5]. Let $\zeta \in \partial\mathbb{D}$. Then, the conjugation ρ_ζ of the rotation $z \mapsto \zeta z$ by T_p is an analytic automorphism of \mathbb{D} contained in \mathcal{B}_p . More explicitly, ρ_ζ is expressed by

$$\rho_\zeta(z) = T_p(\zeta T_p(z)) = \frac{(\zeta - p^2)z + (1 - \zeta)p}{-(1 - \zeta)pz + 1 - p^2\zeta} = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \dots,$$

where

$$(1.2) \quad \alpha_0 = \frac{(1 - \zeta)p}{1 - p^2\zeta}, \quad \text{and} \quad \alpha_k = \frac{\zeta(1 - p^2)^2(1 - \zeta)^{k-1}p^{k-1}}{(1 - p^2\zeta)^{k+1}}, \quad k = 1, 2, 3, \dots$$

Obviously, ρ_ζ can be defined for $\zeta \in \overline{\mathbb{D}}$ as an analytic endomorphism of \mathbb{D} . Noting the fact that

$$\frac{-2\rho_\zeta(z)}{1 - z\rho_\zeta(z)} = \frac{(\zeta(z - p)^2 - (1 - pz)^2)'}{\zeta(z - p)^2 - (1 - pz)^2},$$

we see that the function determined by (1.1) with the choice $\varphi = \rho_\zeta$ is given by

$$(1.3) \quad \begin{aligned} F_\zeta(z) &= \frac{z - T_p(p\zeta)z^2}{(1 - z/p)(1 - pz)} \\ &= \frac{z}{1 - p^2\zeta} \left[\frac{1}{1 - z/p} - \frac{p^2\zeta}{1 - pz} \right] \\ &= \sum_{n=1}^{\infty} \frac{1 - p^{2n}\zeta}{p^{n-1}(1 - p^2\zeta)} z^n =: \sum_{n=1}^{\infty} A_n(\zeta) z^n. \end{aligned}$$

Thus we see that the coefficient region $\{a_n(f) : f \in \mathcal{C}o_p\}$ contains the set $A_n(\overline{\mathbb{D}}) = \{A_n(\zeta) : \zeta \in \overline{\mathbb{D}}\}$. We note that $A_n(\overline{\mathbb{D}})$ is the closed disk $|w - (1 - p^{2n+2})/p^{n-1}(1 - p^4)| \leq (p^2 - p^{2n})/p^{n-1}(1 - p^4)$. Indeed, Avkhadiiev and Wirths [4] proved the following.

Theorem B. *Let $0 < p < 1$ and $n \geq 2$. Then*

$$\{a_n(f) : f \in \mathcal{C}o_p\} = A_n(\overline{\mathbb{D}}) = \left\{ w : \left| w - \frac{1 - p^{2n+2}}{p^{n-1}(1 - p^4)} \right| \leq \frac{p^2 - p^{2n}}{p^{n-1}(1 - p^4)} \right\}.$$

Moreover, for $f \in \mathcal{C}o_p$, $a_n(f) \in \partial A_n(\overline{\mathbb{D}})$ if and only if $f = F_\zeta$ for some $\zeta \in \partial\mathbb{D}$.

Note that for each $\zeta \in \partial\mathbb{D}$, $T_p(p\zeta) = (1 + e^{i\theta})p/(1 + p^2)$ for some $\theta \in \mathbb{R}$ and vice versa. In the present paper, we consider the second Hankel determinant of order 2 for $f(z) = z + a_2 z^2 + \dots$, which is defined by

$$H(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

Especially, we will take a closer look at the variability region $H(\mathcal{C}o_p) = \{H(f) : f \in \mathcal{C}o_p\}$ for $0 < p < 1$. The second Hankel determinant of general order was studied by Pommerenke [12] and Hayman [9] and many others in recent years. A straightforward computation yields

$$H(F_\zeta) = A_2(\zeta)A_4(\zeta) - A_3(\zeta)^2 = -\frac{(1-p^2)^2\zeta}{(1-p^2\zeta)^2} = -\frac{(1-p^2)^2}{p^2}K(p^2\zeta),$$

where

$$K(z) = \frac{z}{(1-z)^2}$$

is the Koebe function. Set

$$(1.4) \quad \Omega_p = \{H(F_\zeta) : |\zeta| \leq 1\} = -\frac{(1-p^2)^2}{p^2}K(p^2\overline{\mathbb{D}}).$$

This set has the following property.

Proposition 1.1. $\Omega_p \subset \Omega_q$ for $0 < q < p < 1$ and

$$\bigcup_{0 < p < 1} \Omega_p = \mathbb{D} \cup \{-1\} \quad \text{and} \quad \bigcap_{0 < p < 1} \Omega_p = \{-(1+z)^2/4 : |z| \leq 1\}.$$

Note that the set $\{-(1+z)^2/4 : |z| \leq 1\}$ is a closed Jordan domain, bounded by a cardioid with an inward-pointing cusp at the origin.

By the above observations, we have $\Omega_p \subset H(\mathcal{C}o_p)$. In view of the coefficient regions of a_n for $\mathcal{C}o_p$, one might suspect that $H(\mathcal{C}o_p) = \Omega_p$ for $0 < p < 1$ and, in particular, $H(\mathcal{C}o_p) \subset \overline{\mathbb{D}}$. This is, however, not the case. To state our result, we set

$$M(p) = \sup\{|H(f)| : f \in \mathcal{C}o_p\}.$$

Theorem 1.2. *Let $0 < p < 1$. Then $M(p) > 1$. Moreover,*

$$\frac{1}{3p} < M(p) < \frac{1}{3p} + \frac{2}{3}.$$

In Section 3, we will prove the above proposition and the theorem. Indeed, we give a description of the variability region of $H(f)$ for $f \in \mathcal{C}o_p$ in Proposition 3.1 below. As a preliminary, we give an explicit form of the coefficient body of order 2 for the class \mathcal{B}_p in Section 2. Our basic idea is to employ an higher-order analogue of Dieusonné's lemma.

2. HIGHER-ORDER ANALOGUE OF DIEUDONNÉ'S LEMMA AND ITS APPLICATION

We expand a function $\varphi \in \mathcal{B}_p$ in the form

$$(2.1) \quad \varphi(z) = c_0 + c_1z + c_2z^2 + \dots, \quad |z| < 1.$$

Then the early coefficients of the function $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ determined by (1.1) are given by

$$(2.2) \quad \begin{aligned} a_2 &= P - c_0, \\ a_3 &= P^2 + \frac{-c_1 + c_0^2 - 4Pc_0 - 2}{3}, \\ a_4 &= P^3 + \frac{-c_2 + c_0 c_1 + 6c_0 - 9P - 9P^2 c_0 + 3Pc_0^2 - 3Pc_1}{6}, \end{aligned}$$

where we put

$$P = p + \frac{1}{p} = \frac{1 + p^2}{p}.$$

By making use of this type of relations, a coefficient problem for $\mathcal{C}o_p$ reduces in principle to that of \mathcal{B}_p . Based on this idea, in [11], the authors solved the extremal problem on $|a_3 - \mu a_2^2|$ for a function $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ in $\mathcal{C}o_p$ and a real constant μ . The key ingredient in [11] was the determination of the coefficient body $\mathbf{X}_1(\mathcal{B}_p)$ of order 1 for \mathcal{B}_p . Here, for $n \geq 0$, the coefficient body $\mathbf{X}_n(\mathcal{F})$ of order n for a class \mathcal{F} of analytic functions at the origin is defined by

$$\begin{aligned} \mathbf{X}_n(\mathcal{F}) &= \{(c_0, c_1, \dots, c_n) \in \mathbb{C}^{n+1} : \\ &\quad \varphi(z) = c_0 + c_1 z + \dots + c_n z^n + O(z^{n+1}) \text{ for some } \varphi \in \mathcal{F}\}. \end{aligned}$$

The goal of the present section is to show the following description of the coefficient body $\mathbf{X}_2(\mathcal{B}_p)$ of order 2.

Theorem 2.1. *Let $0 < p < 1$. A triple (c_0, c_1, c_2) of complex numbers is contained in the coefficient body $\mathbf{X}_2(\mathcal{B}_p)$ if and only if*

$$c_0 = P^{-1}(1 - \sigma_0), \quad \text{and} \quad c_1 = P^{-2}[1 + (P^2 - 2)\sigma_0 + \sigma_0^2] + P^{-1}(1 - |\sigma_0|^2)\sigma_1$$

and

$$\begin{aligned} c_2 &= P^{-3}(1 - \sigma_0)[1 + (P^2 - 2)\sigma_0 + \sigma_0^2] - P^{-2}(P^2 - 2 + 2\sigma_0)(1 - |\sigma_0|^2)\sigma_1 \\ &\quad + \varepsilon P^{-1}(1 - |\sigma_0|^2)\overline{\sigma_0}\sigma_1^2 + P^{-1}(1 - |\sigma_0|^2)(1 - |\sigma_1|^2)\sigma_2 \end{aligned}$$

for some $\sigma_0, \sigma_1, \sigma_2 \in \overline{\mathbb{D}}$, where $P = (1 + p^2)/p > 2$ and $\varepsilon = |1 + p^2\sigma_0|/(1 + p^2\sigma_0) \in \partial\mathbb{D}$.

Let us now recall Dieudonné's lemma (see, for instance, [8, p. 198]).

Lemma 2.2 (Dieudonné's lemma). *Let $z_0, \tau_0 \in \mathbb{D}$ with $|\tau_0| \leq |z_0| \neq 0$. Then the variability region of $\tau_1 = \psi'(z_0)$ for $\psi \in \text{End}(\mathbb{D})$ with $\psi(0) = 0, \psi(z_0) = \tau_0$ is the closed disk given by*

$$(2.3) \quad \left| \tau_1 - \frac{\tau_0}{z_0} \right| \leq \frac{|z_0|^2 - |\tau_0|^2}{|z_0|(1 - |z_0|^2)}.$$

We remark that equality holds in Dieudonné's lemma if and only if ψ is a finite Blaschke product of degree at most 2 (cf. [7, Theorem 3.6]). The following result can be regarded as Dieudonné's lemma of the second order (see [7, Theorem 3.7]).

Lemma 2.3. *Let $z_0, \tau_0 \in \mathbb{D}$ with $|\tau_0| < |z_0| \neq 0$ and suppose that $\tau_1 \in \mathbb{C}$ satisfies (2.3). Then the variability region of $\tau_2 = \psi''(z_0)/2!$ for $\psi \in \text{End}(\mathbb{D})$ with $\psi(0) = 0, \psi(z_0) = \tau_0$ and $\psi'(z_0) = \tau_1$ is the closed disk described by*

$$\left| \tau_2 - \frac{\tau_1 - \tau_0/z_0}{z_0(1 - |z_0|^2)} + \frac{\overline{\tau_0}(\tau_1 - \tau_0/z_0)^2}{|z_0|^2 - |\tau_0|^2} \right| + \frac{|z_0||\tau_1 - \tau_0/z_0|^2}{|z_0|^2 - |\tau_0|^2} \leq \frac{|z_0|(1 - |\tau_0/z_0|^2)}{(1 - |z_0|^2)^2}.$$

We remark that equality holds precisely when ψ is a finite Blaschke product of degree at most 3. In [7, Theorem 3.7], the above inequality is stated as a necessary condition. For sufficiency, a construction is given in the proof below.

In order to prove Theorem 2.1, we show a preliminary form of the characterization of $X_2(\mathcal{B}_p)$.

Lemma 2.4. *Let $0 < p < 1$. A triple (c_0, c_1, c_2) of complex numbers is contained in the coefficient body $\mathbf{X}_2(\mathcal{B}_p)$ if and only if*

$$c_0 = \frac{p - pw_0}{1 - p^2w_0} \quad \text{and} \quad c_1 = \frac{(1 - p^2)^2w_0 + p(1 - p^2)(1 - |w_0|^2)w_1}{(1 - p^2w_0)^2}$$

and

$$c_2 = \frac{(1 - p^2)}{(1 - p^2w_0)^3} [p(1 - p^2)(1 - w_0)w_0 - (1 - p^2)(1 + p^2w_0)(1 - |w_0|^2)w_1 \\ + p(\overline{w_0} - p^2)(1 - |w_0|^2)w_1^2 + p(1 - p^2w_0)(1 - |w_0|^2)(1 - |w_1|^2)w_2]$$

for some $w_0, w_1, w_2 \in \overline{\mathbb{D}}$.

Proof. When $\varphi = \rho_\zeta$ for some $\zeta \in \partial\mathbb{D}$, the coefficients $\alpha_0, \alpha_1, \alpha_2$ are given as c_0, c_1, c_2 with $(w_0, w_1, w_2) = (\zeta, 0, 0)$ (see (1.2)).

We next suppose that a function $\varphi(z) = c_0 + c_1z + c_2z^2 + \dots$ in \mathcal{B}_p is not of the form ρ_ζ , $\zeta \in \partial\mathbb{D}$. Then $\psi = T_p \circ \varphi \circ T_p \in \text{End}(\mathbb{D})$ satisfies $\psi(0) = 0$ but is not a rotation about 0. It is straightforward to check the formulae:

$$\begin{aligned} \tau_0 &:= \psi(p) = T_p(c_0) = \frac{p - c_0}{1 - pc_0}, \\ \tau_1 &:= \psi'(p) = \frac{c_1}{(1 - pc_0)^2}, \\ \tau_2 &:= \frac{\psi''(p)}{2} = \frac{-(1 - pc_0)c_2 + pc_1(1 - pc_0 - c_1)}{(1 - p^2)(1 - pc_0)^3}. \end{aligned}$$

Hence,

$$\begin{aligned} (2.4) \quad c_0 &= T_p(\tau_0) = \frac{p - \tau_0}{1 - p\tau_0}, \quad (1 - pc_0)(1 - p\tau_0) = 1 - p^2, \\ c_1 &= (1 - pc_0)^2\tau_1 = \frac{(1 - p^2)^2\tau_1}{(1 - p\tau_0)^2}, \\ c_2 &= \frac{-(1 - p^2)^3(1 - p\tau_0)\tau_2 + p(1 - p^2)^2\tau_1(1 - p\tau_0 - \tau_1 + p^2\tau_1)}{(1 - p\tau_0)^3}. \end{aligned}$$

By Schwarz's lemma and Dieudonné's lemma (with $z_0 = p$), we have
(2.5)

$$w_0 := \frac{\tau_0}{p} \in \mathbb{D} \quad \text{and} \quad w_1 := \frac{\tau_1 - \tau_0/p}{(p^2 - |\tau_0|^2)/p(1 - p^2)} = \frac{(1 - p^2)(\tau_1 - w_0)}{p(1 - |w_0|^2)} \in \overline{\mathbb{D}}.$$

When $|w_1| = 1$, by the remark after Lemma 2.3, $\psi(z)$ is of the form $z\omega([z, p])$, where $\omega(z) = [w_1 z, -w_0]$. Then the first three Taylor coefficients of $\psi(z)$ about $z = 0$ are given by c_0, c_1, c_2 in (1.2) with $w_2 = 0$.

Finally, suppose that $|w_1| < 1$. Then, by Lemma 2.3, we see that

$$(2.6) \quad \begin{aligned} w_2 &:= \left(\tau_2 - \frac{\tau_1 - \tau_0/p}{p(1 - p^2)} + \frac{\overline{\tau_0}(\tau_1 - \tau_0/p)^2}{p^2 - |\tau_0|^2} \right) \div \left(\frac{p(1 - |\tau_0/p|^2)}{(1 - p^2)^2} - \frac{p|\tau_1 - \tau_0/p|^2}{p^2 - |\tau_0|^2} \right) \\ &= \frac{(1 - p^2)^2 \tau_2 - (1 - |w_0|^2)(1 - p\overline{w_0}w_1)w_1}{p(1 - |w_0|^2)(1 - |w_1|^2)} \in \overline{\mathbb{D}}. \end{aligned}$$

Here, note that the denominator does not vanish because of $|w_1| < 1$. Conversely, for $w_0, w_1 \in \mathbb{D}$ and $w_1 \in \overline{\mathbb{D}}$, the function $\psi(z) = z\omega([z, p])$ fulfills the relations in (2.5) and (2.6), where $\omega(z) = [z[w_2 z, -w_1], -w_0]$. (Note that this construction shows the sufficiency part of Lemma 2.3.) We now obtain

$$\begin{aligned} \tau_0 &= pw_0, \\ \tau_1 &= w_0 + \frac{p(1 - |w_0|^2)w_1}{1 - p^2}, \\ \tau_2 &= \frac{1 - |w_0|^2}{(1 - p^2)^2} [(1 - p\overline{w_0}w_1)w_1 + p(1 - |w_1|^2)w_2]. \end{aligned}$$

Substitution of these expressions into (2.4) proves the lemma. \square

Proof of Theorem 2.1. For $w_0, w_1, w_2 \in \overline{\mathbb{D}}$, we put

$$\sigma_0 = [w_0, p^2] = \frac{w_0 - p^2}{1 - p^2 w_0}, \quad \sigma_1 = \frac{|1 - p^2 w_0|^2}{(1 - p^2 w_0)^2} w_1, \quad \sigma_2 = \frac{|1 - p^2 w_0|^2}{(1 - p^2 w_0)^2} w_2.$$

Then $\sigma_j \in \overline{\mathbb{D}}$ for $j = 0, 1, 2$ and vice versa. Noting the elementary relations

$$w_0 = [\sigma_0, -p^2] = \frac{\sigma_0 + p^2}{1 + p^2 \sigma_0}, \quad (1 + p^2 \sigma_0)(1 - p^2 w_0) = 1 - p^4$$

and

$$(1 - |\sigma_0|^2)|1 - p^2 w_0|^2 = (1 - p^4)(1 - |w_0|^2),$$

the formulae of c_j in Lemma 2.4 can be expressed in terms of $\sigma_0, \sigma_1, \sigma_2$ through tedious but straightforward computations. We finally replace $p + 1/p$ by P to prove Theorem 2.1. \square

3. PROOF OF MAIN RESULTS

By the relations (2.2), we can express $H(f)$ for $f \in \mathcal{C}o_p$ in terms of c_j 's as follows:

$$(3.1) \quad 18H(f) = 3(c_0 - P)c_2 - 2c_1^2 + (c_0^2 - 4Pc_0 + 3P^2 - 8)c_1 \\ - (c_0^2 - Pc_0 + 1)(2c_0^2 - 5Pc_0 + 3P^2 + 8).$$

We further substitute the formulae in Theorem 2.1 into (3.1) to obtain

$$(3.2) \quad 18P^3H(f) = -18P[1 + (P^2 - 2)\sigma_0 + \sigma_0^2] \\ + 3[1 - 7P^2 + 2P^4 + (3P^2 - 2)\sigma_0 + \sigma_0^2](1 - |\sigma_0|^2)\sigma_1 \\ + P[2(1 - |\sigma_0|^2) + 3\varepsilon\overline{\sigma_0}(P^2 - 1 + \sigma_0)](1 - |\sigma_0|^2)\sigma_1^2 \\ - 3P(P^2 - 1 + \sigma_0)(1 - |\sigma_0|^2)(1 - |\sigma_1|^2)\sigma_2 \\ =: \Phi_p(\sigma_0, \sigma_1, \sigma_2),$$

where $\varepsilon = |1 + p^2\sigma_0|/(1 + p^2\sigma_0)$. At this stage, we have obtained the following description of the set $H(\mathcal{C}o_p)$.

Proposition 3.1. *Let $0 < p < 1$. Then the variability region of the second Hankel determinant $H(f)$ of order 2 for $f \in \mathcal{C}o_p$ is given by*

$$H(\mathcal{C}o_p) = \{\Phi_p(\sigma_0, \sigma_1, \sigma_2)/18P^3 : \sigma_0, \sigma_1, \sigma_2 \in \overline{\mathbb{D}}\}.$$

We note that the function F_ζ given in (1.3) corresponds to the parameters $(\sigma_0, \sigma_1, \sigma_2) = ([\zeta, p^2], 0, 0)$. Since $\Phi_p(\sigma, 0, 0) = -18P(1 + (P^2 - 2)\sigma + \sigma^2)$, as a by-product, we have the following description of the set Ω_p defined by (1.4).

Lemma 3.2.

$$\Omega_p = \{-P^{-2}[1 + (P^2 - 2)\sigma + \sigma^2] : \sigma \in \overline{\mathbb{D}}\},$$

where $P = (1 + p^2)/p$, $0 < p < 1$.

The description of the Lemma can now be used to show Proposition 1.1.

Proof of Proposition 1.1. Put $t = P^2 > 4$ and write

$$f_t(z) = -t^{-1}[1 + (t - 2)z + z^2].$$

Then $\Omega_p = f_t(\overline{\mathbb{D}})$ by Lemma 3.2. To show the monotonicity of Ω_p , it is enough to prove that $f_t(\mathbb{D}) \subset f_{t'}(\mathbb{D})$ for $4 < t < t'$. We note that $f_t(z)$ is univalent for each $t > 4$. This is implied by the elementary fact that $f(z) = z + az^2$ is univalent (indeed, starlike) if and only if $|a| \leq 1/2$. Hence, $\gamma_t(\theta) = f_t(e^{i\theta})$, $0 \leq \theta \leq 2\pi$, gives a smooth parametrization of the boundary curve of the Jordan domain Ω_p . We first observe that $f_t(1) = -1$ is stationary with respect to t . In order to show that $f_t(\mathbb{D})$ is an increasing family of domains, it is enough to see that the flow $t \mapsto \gamma_t(\theta)$ is outgoing from $f_{t_0}(\mathbb{D})$ at the time $t = t_0$ for each $\theta \in (0, 2\pi)$. Since an outer normal vector of the boundary curve $\partial\Omega_p$ at $\gamma_t(\theta)$ is given by $\gamma'_t(\theta)/i$,

we should show that $|\arg[i\dot{\gamma}_t(\theta)/\gamma'_t(\theta)]| < \pi/2$, where $\dot{\gamma}_t(\theta) = \partial\gamma_t(\theta)/\partial t$. By a straightforward computation, we obtain

$$\operatorname{Re} \frac{\gamma'_t(\theta)/i}{\dot{\gamma}_t(\theta)} = \operatorname{Re} \frac{-t^{-1}(t-2+2e^{i\theta})e^{i\theta}}{t^{-2}(1-e^{i\theta})^2} = \frac{t(t-2+2\cos\theta)}{4\sin^2(\theta/2)} > 0$$

for $0 < \theta < 2\pi$. Thus we have shown the monotonicity of Ω_p in p . The other assertion easily follows from the facts that $\lim_{t \rightarrow 4} f_t(z) = -(1+z)^2/4$ and that $\lim_{t \rightarrow +\infty} f_t(z) = -z$. \square

Finally, we prove our main result.

Proof of Theorem 1.2. Considering $\sigma_0 = t, \sigma_1 = -1$ and $\sigma_2 = 0$ with $t \in [0, 1]$ in Proposition 3.1 above, we obtain

$$\begin{aligned} \Phi_p(t, -1, 0) = & -18P[1 + (P^2 - 2)t + t^2] \\ & - 3[1 - 7P^2 + 2P^4 + (3P^2 - 2)t + t^2](1 - t^2) \\ & + P[2(1 - t^2) + 3t(P^2 - 1 + t)](1 - t^2). \end{aligned}$$

Setting $h_p(t) := -\Phi_p(t, -1, 0)/18P^3$ gives

$$\begin{aligned} 18P^3 h_p(t) = & -(P+3)t^4 - (3P^3 + 9P^2 - 3P - 6)t^3 - (6P^4 - 21P^2 - 17P)t^2 \\ & + 3(7P^3 + 3P^2 - 13P - 2)t + (6P^4 - 21P^2 + 20P + 3) \end{aligned}$$

and

$$\begin{aligned} 18P^3 h'_p(t) = & -4(P+3)t^3 - 3(3P^3 + 9P^2 - 3P - 6)t^2 \\ & + 2(6P^4 - 21P^2 - 17P)t + 3(7P^3 + 3P^2 - 13P - 2), \end{aligned}$$

which leads to $h_p(1) = 1$ and $h'_p(1) = -2(P-2)(P+1)/3P < 0$. Thus the function $h_p(t)$ is strictly decreasing at $t = 1$ and therefore $h_p(t_0) > 1$ for $t_0 < 1$ sufficiently close to 1. Hence, $|H(f)| = h_p(t_0) > 1$ for the function $f \in \mathcal{C}o_p$ corresponding to the parameter triple $(t_0, -1, 0)$. Thus $M(p) > 1$ follows.

To show the inequality $M(p) > 1/3p$, we use the lower estimate

$$M(p) \geq h_p\left(\frac{7}{4P}\right) = \frac{P}{3} + g\left(\frac{1}{P}\right) = \frac{1}{3p} + \frac{p}{3} + g\left(\frac{1}{P}\right),$$

where

$$g(x) = -\frac{7x}{48} + \frac{143x^2}{72} - \frac{121x^3}{128} - \frac{427x^4}{1152} + \frac{343x^5}{384} + \frac{5831x^6}{4608} - \frac{2401x^7}{1536}.$$

We note that $p/3 > 1/3P$. It is not difficult to see that $1/3P + g(1/P) = x/3 + g(x) > 0$ for $x = 1/P \in (0, 1/2)$. Therefore, $M(p) > 1/3p$.

Finally, we show $M(p) < (1+2p)/3p$. In view of (3.2), one can estimate Φ_p as in

$$\begin{aligned} |\Phi_p(\sigma_0, \sigma_1, \sigma_2)| & \leq B_0 + B_1x + B_2x^2 + B_3(1-x^2) \\ & = (B_2 - B_3)x^2 + B_1x + B_0 + B_3 \end{aligned}$$

with $x = |\sigma_1|$, where

$$\begin{aligned} B_0 &= 18P[1 + (P^2 - 2)y + y^2], \\ B_1 &= 3[1 - 7P^2 + 2P^4 + (3P^2 - 2)y + y^2](1 - y^2), \\ B_2 &= P[2(1 - y^2) + 3y(P^2 - 1 + y)](1 - y^2), \\ B_3 &= 3P(P^2 - 1 + y)(1 - y^2) \end{aligned}$$

with $y = |\sigma_0|$. Since

$$[2(1 - y^2) + 3y(P^2 - 1 + y)] - 3P(P^2 - 1 + y) = (1 - y)(1 - y - 3P^2) \leq 0$$

for $P > 2$, $0 \leq y \leq 1$, we have $B_2 - B_3 \leq 0$. Thus, we have

$$\begin{aligned} |\Phi_p(\sigma_0, \sigma_1, \sigma_2)| &\leq B_0 + B_1 + B_3 \\ &= 18P^3 + 6P^2(P^2 - P - 2)2t \\ &\quad - 3P(2P^3 + P^2 + 2P - 4)t^2 + 3(3P^2 + P + 2)t^3 - 3t^4, \end{aligned}$$

where $t = 1 - y \in [0, 1]$.

Using the inequalities $2t \leq 1 + t^2$ and $6t^3 - 3t^4 \leq 3t^2$ for $0 \leq t \leq 1$, we obtain

$$\begin{aligned} B_0 + B_1 + B_3 &\leq 6(P^4 + 2P^3 - 2P^2) + 3(-3P^3 - 6P^2 + 4P + 1)t^2 \\ &\quad + 3P(3P + 1)t^3 =: G_p(t). \end{aligned}$$

The function $G_p(t)$ has a maximum at $t = 0$ (and a minimum at $t = 2(3P^3 + 6P^2 - 4P - 1)/(3P(3P + 1)) > 1$) for all $P > 2$. Therefore we have

$$\sup_{\sigma_0, \sigma_1, \sigma_2 \in \mathbb{D}} |\Phi_p(\sigma_0, \sigma_1, \sigma_2)| \leq \max_{0 \leq t \leq 1} G_p(t) = G_p(0) = 6P^2(P^2 + 2P - 2),$$

which implies according to Proposition 3.1

$$M(p) \leq \frac{P^2 + 2P - 2}{3P} = \frac{1 + 2p}{3p} - \frac{p(1 - p^2)}{1 + p^2} < \frac{1}{3p} + \frac{2}{3}.$$

This completes the proof. \square

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